L36 April 16 Fixed Pt

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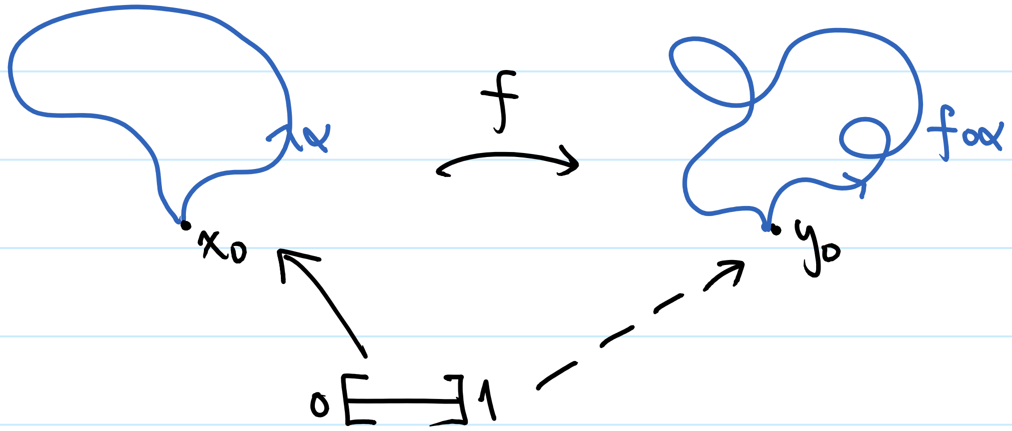
Recall several facts

Theorem Let X, Y be path connected
$f:(X,x_0)\longrightarrow(Y,y_0)$ be continuous
Then $f_\#:\pi_1(X,x_0)\longrightarrow\pi_1(Y,y_0)$ is
a homomorphism where

$$\alpha \overset{f_\#}{\longmapsto} f\circ\alpha$$



Obviously, we used $\alpha_0\simeq\alpha_1 \Longrightarrow f\circ\alpha_0\simeq f\circ\alpha_1$
and $f\circ(\alpha*\beta)=(f\circ\alpha)*(f\circ\beta)$

Fact. $\pi_1(S^1)=\mathbb{Z}$, $\pi_1(\mathbb{D}^2)$ is trivial
where $S^1=\{|z|=1\}\subset\mathbb{D}^2=\{|z|\leq 1\}$

Theorem. S^1 is never a retract of $\mathbb{D}^2$

The proof makes use of π_1

In general, S^{n-1} is never a retract of $\mathbb{D}^n$
by using high dimensional algebraic topology

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Suppose $\exists \; r : \mathbb{D}^2 \longrightarrow S^1$ such that

$$\begin{tikzcd}
S^1 \arrow[r, hook, "i"] \arrow[rr, bend right, "r \circ i = id_{S^1}"'] & \mathbb{D}^2 \arrow[r, "r"] & S^1
\end{tikzcd}$$

Then

$$\begin{tikzcd}
\pi_1(S^1) \arrow[r, "i_\#"] \arrow[rr, bend right] & \pi_1(\mathbb{D}^2) \arrow[r, "r_\#"] & \pi_1(S^1)
\end{tikzcd}$$

$$r_\# \circ i_\# \underset{\uparrow}{=} (r \circ i)_\# = id$$

Exercise

$$\begin{tikzcd}
(\mathbb{Z}, +) \arrow[r, "i_\#"] \arrow[rr, bend right, "id"'] & (0, +) \arrow[r, "r_\#"] & (\mathbb{Z}, +)
\end{tikzcd}$$

$$\begin{tikzcd}
n \neq 0 \quad 1 \arrow[r, mapsto, "i_\#"] \arrow[drr, bend right, "id"'] & 0 \arrow[r, mapsto, "r_\#"] & 0 \arrow[d, phantom, "\neq"] \\
& & n
\end{tikzcd}$$

Brouwer Fixed Point Theorem

Any continuous map $f : \mathbb{D}^n \longrightarrow \mathbb{D}^n$ must have a fixed point,

Suppose not, then $\forall \; x \in \mathbb{D}^n \;\; f(x) \neq x$

$\exists$ straight line $L_x = \{ (1-t)x + t f(x) : t \in \mathbb{R} \}$

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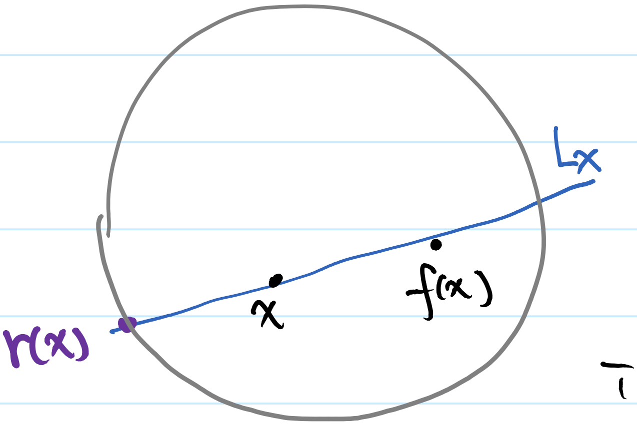


Let
$$\begin{gathered} r(x) \\ \| \\ (1-t)x + tf(x) \end{gathered}$$
with $|r(x)| = 1$ and $t \lesssim 0$

Then $r: D^n \longrightarrow S^{n-1}$

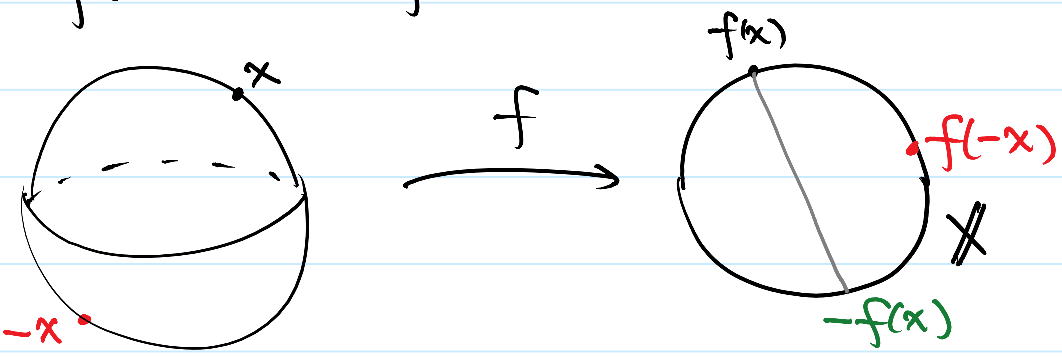
with $r|_{S^{n-1}} = \mathrm{id}_{S^{n-1}}$

This makes S^{n-1} a retract of D^n Contradiction

Borsuk-Ulam Theorem

There is no continuous $f: S^n \longrightarrow S^{n-1}$

with $f(-x) = -f(x)$



The proof makes use of projective space

For $n = 2$, assume there exists such f

$$\begin{tikzcd} S^2 \arrow[r, "f"] \arrow[d, "q"'] & S^1 \arrow[d, "q"] \\ \mathbb{P}^2 \arrow[r, "\hat{f}"] & \mathbb{P}^1 = S^1 \end{tikzcd}$$

Then $\hat{f}$ is defined

$$\hat{f}[x] = [f(x)]$$

$f(x) \longmapsto f(x)$

$f(-x) \longmapsto -f(x)$

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$$\begin{tikzcd}
(0,+) = \pi_1(S^2) \arrow[r, "f_\#"] \arrow[d, "q_\#"'] & \pi_1(S^1) = (\mathbb{Z},+) \arrow[d, "q_\#"] \\
(\mathbb{Z}/2,+) = \pi_1(\mathbb{P}^2) \arrow[r, "\hat{f}_\#"] & \pi_1(\mathbb{P}^1) = \pi_1(S^1) = (\mathbb{Z},+)
\end{tikzcd}$$

By Algebra, $\hat{f}_\# \equiv 0$ (Exercise)

But we can create a loop $\alpha : [0,1] \longrightarrow \mathbb{P}^2$ which has $\hat{f} \circ \alpha$ non-trivial in S^1

Hairy-Ball Theorem

Every vector field on S^{2n} has a zero.

Ham-Sandwich Theorem

Let $A_1, A_2, \cdots, A_n$ be bounded measurable sets in $\mathbb{R}^n$. Then $\exists$ hyperplane which divide all A_j into half measure.